CONVECTIVE DIFFUSION OF SALTS IN A RADIAL FLOW OF GROUNDWATER

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This type of motion occurs, for example, when industrial effluent is pumped into aquifers via boreholes. The results for these may sometimes be extended to infiltration from sumps, storage tanks, and so on.

This motion is described by the differential equation

$$\frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right)-V\frac{\partial C}{\partial r}=n\frac{\partial C}{\partial t}. \qquad (1)$$

Here C is the salt concentration at a point r at time t, V is the filtration rate, n is the porosity of the rock, and D is diffusion coefficient.

Recent studies [1, 2] have shown that D is substantially dependent on V and on the properties of the rock and liquid, so it may be taken as a parameter characterizing the properties of the salt solution together with the hydrodynamic features of the rock. It is therefore called the coefficient of infiltration diffusion; numerical values are best found from field and laboratory tests under convective conditions.

The V for the quasi-steady state is given by

$$V=\frac{Q_w}{2\pi hr}, \qquad (2)$$

in which Q_w is the flow rate, h is the thickness of the aquifer, and r is radius.

We introduce the function $C^\circ = C - C_0$, in which C_0 is the salt concentration in the aquifer at t = 0. Then (1) and (2) give

$$\frac{\partial^2 C^\circ}{\partial r^2}+\frac{1}{r}(1-2\nu)\frac{\partial C^\circ}{\partial r}=\frac{1}{D^\circ}\frac{\partial C^\circ}{\partial t},$$

$$\nu=\frac{Q_w}{4\pi hD^\circ n},\qquad D^\circ=\frac{D}{n}. \qquad (3)$$

Equation (3) is usually solved subject to a condition of the first kind at $r = r_0$ (r_0 being the radius of the borehole), namely $C^\circ = C_c^\circ =$ = constant [2, 9, 10]; but this condition far from always applies. Rather, a condition of the third kind [3] applies to the interaction of the incoming flow with the groundwater, which is characterized by constancy of the salt flux Q_s:

$$t>0,\qquad r=r_0,\qquad 2\pi hDr\frac{\partial C^\circ}{\partial r}-Q_wC^\circ=-Q_s=\text{const},$$

$$Q_s=Q_wC_k^\circ, \qquad (4)$$

in which $C_k^\circ = C_k - C_0$, C_k being the salt concentration in the water pumped into the borehole.

We apply (4) to an unbounded region, i.e., we put

$$t>0,\qquad r\to\infty,\qquad \frac{\partial C^\circ}{\partial r}=0, \qquad (5)$$

which enables us to solve (3) as follows.

We apply the Laplace transformation

$$C^\circ(r,t)\div T(r,p)=\int_0^\infty C^\circ(r,\tau)\cdot e^{-p\tau}d\tau.$$

Then (3) is replaced by the ordinary differential equation

$$T''+\frac{1}{r}(1-2\nu)T'-\beta^2T=0,\qquad \beta=\left(\frac{p}{D^\circ}\right)^{1/2}, \qquad (6)$$

whose solution subject to (5) is [4]

$$T=Ar^\nu K_\nu(\beta r), \qquad (7)$$

in which K_ν is McDonald's symbol and A is a constant relative to r.

Condition (4) is transformed to

$$r=r_0,\qquad 2\pi hDrT'-Q_wT=-\frac{Q_s}{p}, \qquad (8)$$

which is substituted into (7); A is determined, which gives us

$$T=\frac{Q_sr^\nu K_\nu(\beta r)}{pr_0^\nu\,[2\pi hDr_0\beta K_{\nu-1}(\beta r_0)+Q_wK_\nu(\beta r_0)]}. \qquad (9)$$

The Riemann-Mellin formula takes us from the transform to the original, but the result is cumbersome. A solution more convenient for practical use is obtained if we note that the argument of the functions in the denominator of (9) is small, since r_0 is small relative to the entire infiltration region, and we can assume for t large that $\beta r_0 \ll 1$. In that case the following approximate expressions [4] apply

$$K_\nu(\beta r_0)\approx\frac{1}{2}\Gamma(\nu)\left(\frac{2}{\beta r_0}\right)^\nu,$$

$$K_{\nu-1}(\beta r_0)\approx\frac{1}{2}\Gamma(\nu-1)\left(\frac{2}{\beta r_0}\right)^{\nu-1}, \qquad (10)$$

in which Γ represents the gamma function. This is substituted into (9) to give

$$T\approx\frac{Q_s(r\beta)^\nu K_\nu(\beta r)}{2^{\nu-1}\pi hDr_0^2\Gamma(\nu-1)\,p\,[\beta^2+Q_w(\nu-1)/\pi hDr_0^2]}. \qquad (11)$$

Tabulated relations [5] give us the original, the solution being

$$C(r,t)=C_0+\frac{Q_s}{Q_w}[1-F_1(\lambda,\nu)-F_2(\lambda,\nu,B)],$$

$$\left(\lambda=\frac{r^2}{4D^\circ t}\right) \qquad (12)$$

$$F_1(\lambda,\nu)=\frac{1}{\Gamma(\nu)}\int_0^\lambda e^{-\alpha}\alpha^{\nu-1}d\alpha, \qquad (13)$$

$$F_2(\lambda,\nu,B)=\frac{1}{\Gamma(\nu)}\exp\frac{-B}{\lambda}\int_\lambda^\infty e^{-\alpha+\frac{B}{\alpha}}\alpha^{\nu-1}d\alpha, \qquad (14)$$

$$B=\frac{Q_w}{4\pi hD^\circ n}\left(\frac{r}{r_0}\right)^2(\nu-1),$$

$$F_1(\lambda,\nu)=1\quad\text{for } t=0,$$

$$\lambda=\infty,\ \lambda=0,\ F_1(\lambda,\nu)=0\quad\text{for } t=\infty.$$

Expression (13) is the normalized incomplete gamma function, which has been tabulated in detail for wide ranges of the parameters λ and ν [6].

The $F_2(\lambda, \nu, B)$ of (14) is* given by the following formula, which is useful in calculations (ν is equal to an integer):

$$F_2(\lambda,\nu,B)=\frac{e^{-\frac{B}{\lambda}}}{\Gamma(\nu)}\Bigg[\sum_{k=0}^{\nu-1}\frac{B^k(\nu-k-1)!}{k!}(1-F_1(\lambda,\nu-k)+$$

$$+\sum_{k=\nu}^{\infty}\frac{B^k\lambda^{\nu-k}}{k!}E_{k+1-\nu}(\lambda)\Bigg] \qquad (15)$$

in which $E_m(\lambda)$ is the integral-exponential function [7].

We use Laplace's method of estimating the integral for large values of the parameter [8], which gives us an asymptotic formula for F_2 for B large and $\lambda \ll B$, $\nu \ll B$:

$$F_2(\lambda,\nu,B)\approx\frac{e^{-\lambda}\lambda^{\nu+1}}{\Gamma(\nu)B}\left[\sum_{n=1}^N a_nB^{-n+1}+0(B^{-N+1})\right], \qquad (16)$$

*Expressions (15)–(17) were derived by V. I. Pagurova.

$$a_1 = 1, \quad a_2 = \lambda^2 \left(\frac{\nu + 1}{\lambda} - 1\right),$$

$$a_m = \left(\frac{\nu + 2m - 3}{\lambda} - 1\right) \lambda^2 a_{m-1} - (m - 2)(m + \nu - 2) \lambda^2 a_{m-2},$$

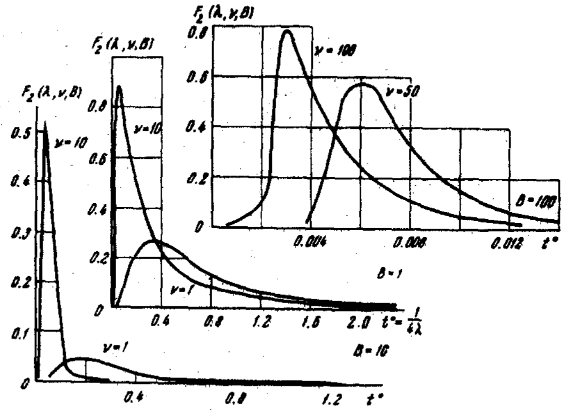
$$m = 3, 4, 5, \ldots \qquad (16) \text{ (cont'd)}$$

There is a recurrence relation with respect to ν:

$$F_2(\lambda, \nu, B) = \frac{e^{-\lambda}\lambda^{\nu-1}}{\Gamma(\nu)} - \frac{B}{(\nu-1)(\nu-2)} F_2(\lambda, \nu-2, B) + F_2(\lambda, \nu-1, B),$$

$$F_2(\lambda, \nu, B) = 0 \quad (t = 0, \ \lambda = \infty), \quad (t = \infty, \ \lambda = 0). \qquad (17)$$

The figure shows $F_2(\lambda, \nu, B)$ for some values of λ, ν, and B.



The solution is [9] as follows when C_c (salt concentration at the contour of the well) is constant

$$C(r, t) = C_0 + (C_c - C_0)[1 - F_1(\nu, \lambda)], \qquad (18)$$

in which $F_1(\nu, \lambda)$ is given by (13).

Comparison of (12) and (18) shows that the convective transport in the initial period, when the salt flow rate in the borehole is constant, is characterized by a broader dispersal zone than when the concentration is constant.

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